Class 8, given on Jan 21, 2010, for Math 13, Winter 2010

Recall how we evaluated integrals using polar coordinates: if we have a region D defined in polar coordinates by inequalities $0 \le a \le r \le b, \alpha \le \theta \le \beta$, with $0 \le \beta - \alpha \le 2\pi$ (a so-called *polar rectangle*), then the double integral of a function $f(x, y)$ over D is given by the iterated integral

$$\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

Let's look at a few more examples involving polar coordinates.

Example. Let D be the region bounded by the lines $x = 0, x = y$, and the circle $x^2+y^2 = 9$. Evaluate the double integral

$$\iint_D x\, dA.$$

If we wanted to, we could set up an iterated integral using rectangular coordinates to evaluate this double integral, but let us use polar coordinates instead. The region D can be described with the polar inequalities $0 \le r \le 3, \pi/4 \le \theta \le \pi/2$. Therefore the iterated integral, in polar coordinates, which we want to calculate is

$$\int_{\pi/4}^{\pi/2} \int_0^3 (r\cos\theta) r\, dr\, d\theta = \int_{\pi/4}^{\pi/2} \int_0^3 r^2 \cos\theta\, dr\, d\theta.$$

This integral is easy to evaluate:

$$\int_{\pi/4}^{\pi/2} \left(\frac{r^3}{3} \cos\theta \Big|_{r=0}^{r=3} \right) d\theta = \int_{\pi/4}^{\pi/2} 9\cos\theta\, d\theta = 9\sin\theta \Big|_{\pi/4}^{\pi/2} = 9 - 9/\sqrt{2}.$$

So far, we have discussed evaluating integrals over domains which are polar rectangles; i.e. of the form $a \le r \le b, \alpha \le \theta \le \beta$. However, we can also consider more general domains of the form $r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta$, where the inequalities for r depend on θ. And just like the case with rectangular coordinates, the corresponding iterated integral becomes

$$\int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

Example. Find the area of the region defined by the polar inequalities $0 \le \theta \le 2\pi, 0 \le r \le \theta$. (Notice that this is the area 'enclosed' by one loop of the spiral of Archimedes.)

If we let D be the region defined by the above inequalities, then the area of D is given by the iterated integral

$$\iint_D dA = \int_0^{2\pi} \int_0^\theta r\, dr\, d\theta.$$

Evaluating this iterated integral gives:

$$\int_0^{2\pi} \left(\frac{r^2}{2} \Big|_{r=0}^{r=\theta} \right) d\theta = \int_0^{2\pi} \frac{\theta^2}{2}\, d\theta = \frac{\theta^3}{6} \Big|_0^{2\pi} = \frac{4\pi^3}{3}.$$

Notice that this integral probably would have been very hard to calculate using rectangular coordinates.

Example. Let D be the half-annulus given by the inequalities $1 \leq x^2 + y^2 \leq 4, y \geq 0$. Evaluate the double integral

$$\iint\limits_D \sin(x^2 + y^2)\, dA.$$

By now, it should be fairly clear that a good place to start with this problem is to convert this integral into an iterated integral over polar coordinates, since the domain of integration seems well suited to the use of polar coordinates. (If you try to evaluate this integral using rectangular coordinates you will fail, anyway, since you end up needing to evaluate the integral of $\sin x^2$.)

The region D is described by polar inequalities $1 \leq r \leq 2, 0 \leq \theta \leq \pi$. Therefore, the iterated integral we want to calculate is

$$\int_0^\pi \int_1^2 \sin((r\cos\theta)^2 + (r\sin\theta)^2) r\, dr\, d\theta = \int_0^\pi \int_1^2 r \sin(r^2)\, dr\, d\theta.$$

Notice that the factor of r saves the day again! We evaluate this integral:

$$\int_0^\pi \left(\frac{-\cos(r^2)}{2} \Big|_{r=1}^{r=2} \right) d\theta = \int_0^\pi \frac{\cos 1 - \cos 4}{2}\, d\theta = \frac{\pi(\cos 1 - \cos 4)}{2}.$$

Example. We now consider a very clever use of polar coordinates to evaluate an important integral. Consider the improper integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

We can't evaluate this integral by finding an antiderivative for the integrand since there is no such expression for this antiderivative using any functions we know. Nevertheless, it is possible to calculate the value of this definite integral using integration over polar coordinates and a very clever trick!

This integral is of central importance in probability theory, because it turns out to be very closely related to the standard normal distribution, which is also known as the Gaussian or Bell curve. Therefore, it is very important to know how to calculate the value of this integral.

Since there don't seem to be any double integrals in sight, we manufacture one in the following clever way. Consider I^2, written in the following strange form:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2}\, dy \right).$$

The clever observation is that this product of integrals can be rewritten as a single double integral, over the domain $\mathbb{R}^2$. (We technically have not defined double integrals over unbounded regions, but the idea is similar to how improper integrals are defined. One takes a limit of the value of this double integral taken over larger and larger rectangles.) It is a general fact that the double integral of a function of the form $f(x)g(y)$ over the rectangle $R = [a,b] \times [c,d]$ is equal to

$$\int_a^b \int_c^d f(x)g(y)\, dy\, dx = \left(\int_a^b f(x)\, dx \right) \left(\int_c^d f(x)\, dx \right).$$

You can see this just by evaluating the iterated integral on the left. In any case, using this fact, we can rewrite I^2 as follows:

$$I^2 = \iint\limits_{\mathbb{R}^2} e^{-x^2} e^{-y^2}\, dA = \iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA.$$

At this point, we write this double integral as an iterated integral using polar coordinates. (One might think to do this because of the appearance of $x^2 + y^2$ in the integrand.) First, the domain $D = \mathbb{R}^2$ can be expressed using polar coordinates as $0 \leq r \leq \infty, 0 \leq \theta \leq 2\pi$. Therefore, the integral we want to evaluate is

$$\int_0^{2\pi} \int_0^{\infty} re^{-r^2}\, dr\, d\theta.$$

Notice that the extra factor of r allows us to evaluate this integral:

$$\int_0^{2\pi} \left(\frac{-e^{-r^2}}{2} \Big|_{r=0}^{\infty} \right) d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \frac{2\pi}{2} = \pi.$$

Therefore, the value of the integral which we wanted to originally calculate is $I = \sqrt{\pi}$.